数理解析研究所講究録
第 778 巻 1992 年 84-90

A Construction of Solutions of the Ernst Equations

Takashi HASHIMOTO and Ryuichi SAWAE

Department of Mathematics, Faculty of Science
Hiroshima University, Higashi-Hiroshima, 724, Japan

In this article, we give a prescription for constructing formal solutions of the Ernst equations which are derived from the stationary axially symmetric Einstein-Maxwell equations. This is based on the treatment of [1].

0. Preliminaries

Let $ds^2 = g_{ij}dx^i dx^j$ be a metric and $A = A_i dx^i$ a electro-magnetic potential on $\mathbb{R}^{1+3}$. Then the Einstein-Maxwell field equations are given by

$$R_{ij} = 8\pi T_{ij}, \quad \nabla_k F^{ik} = 0 \quad (i, j, k = 0, 1, 2, 3),$$

where R_{ij} is Ricci curvature and

$$\begin{aligned} F_{ij} &= \partial_i A_j - \partial_j A_i, \\ T_{ij} &= \frac{1}{8\pi}(F_{ik}F_j{}^k - \frac{1}{4}g_{ij}F_{kl}F^{kl}). \end{aligned}$$

Since we are concerned with stationary axisymmetric solutions, we choose a coordinates $(x^0, x^1, x^2, x^3) = (\tau, \phi, z, \rho)$ on $\mathbb{R}^{1+3}$ where τ is *time* and (ϕ, z, ρ) are the cylindrical coordinates on $\mathbb{R}^3$.

We assume that the metric ds^2 takes the form

$$ds^2 = \sum_{i=0}^{1} h_{ij}dx^i dx^j - \lambda^2((dx^1)^2 + (dx^2)^2) \quad (\lambda > 0)$$

and $h = (h_{ij}), \lambda$ and A_i depend only on z and ρ. Moreover, we assume that $h_{00} \neq 0$, det $h = -\rho^2$ and $A_2 = A_3 = 0$, which are physically reasonable.

Then the stationary axisymmetric Einstein-Maxwell field equations are given, in matrix form, as follows:

$$d(\rho^{-1}h\epsilon * dA) = 0 \tag{1}$$

$$d\left\{\rho^{-1}h\epsilon * dh - 2(\rho^{-1}h\epsilon * dA)^t A - 2A^t(\rho^{-1}h\epsilon * dA)\right\} = 0, \tag{2}$$

$$\frac{\partial_z \lambda}{\lambda} = \frac{\rho}{4}\mathrm{tr}(h^{-1}\partial_\rho h h^{-1}\partial_z h) - 2\rho\partial_\rho{}^t A h^{-1}\partial_z A, \tag{3.a}$$

$$\frac{\partial_\rho \lambda}{\lambda} = -\frac{1}{2\rho} + \frac{\rho}{8}\mathrm{tr}\{(h^{-1}\partial_\rho h)^2 - (h^{-1}\partial_z h)^2\}$$
$$- \rho(\partial_\rho {}^t A h^{-1} \partial_\rho A - \partial_z {}^t A h^{-1} \partial_z A), \tag{3.b}$$

where $A = \begin{pmatrix} A_0 \\ A_1 \end{pmatrix}$, $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $*$ =Hodge operator for the metric $dz^2 + d\rho^2$. Since $h_{00} \neq 0$ and $\det h = -\rho^2$, we can parametrize h as

$$h = \begin{pmatrix} f & f\omega \\ f\omega & f\omega^2 - \rho^2/f \end{pmatrix}.$$

It is known that (3.a) and (3.b) are integrable, so we shall be concerned with (1) and (2) in what follows.

Next we introduce the so-called Ernst potential.

Note that every closed form is exact since we consider it locally.

From (1), there exists a 2×1-matrix valued function $B = \begin{pmatrix} B_0 \\ B_1 \end{pmatrix}$ such that

$$*dB = \rho^{-1} h \epsilon dA. \tag{4}$$

Substituting (4) into (2),

$$d(\rho^{-1} h\epsilon * dh + 2dB\,{}^t A + 2A d\,{}^t B) = 0.$$

The (1,1)-th entry reads

$$d(\rho^{-1} f^2 * d\omega + 2A_0 dB_0 - 2B_0 dA_0) = 0.$$

Therefore, there exists ψ such that

$$\rho^{-1} f^2 d\omega = *d\psi + 2(A_0 * dB_0 - B_0 * dA_0) = 0.$$

Using f, A_0, b_0 and ψ, we put

$$v = A_0 + iB_0, \quad u = f - |v|^2 + i\psi.$$

The pair (u, v) is called the Ernst potential. Then the following fact is well known.

PROPOSITION 1. *(h, A) is a solution of (1) and (2) if and only if (u, v) is a solution of the following equations:*

$$f(d * du + \rho^{-1} d\rho \wedge *du) = (du + 2\bar{v}dv) \wedge *du, \tag{5}$$
$$f(d * dv + \rho^{-1} d\rho \wedge *dv) = (du + 2\bar{v}dv) \wedge *dv. \tag{6}$$

But we change the definition of u into the following one:

$$u = f + |v|^2 + i\psi,$$

so that our Ernst equations become

$$f(d * du + \rho^{-1} d\rho \wedge *du) = (du - 2\bar{v}dv) \wedge *du, \tag{5'}$$
$$f(d * dv + \rho^{-1} d\rho \wedge *dv) = (du - 2\bar{v}dv) \wedge *dv. \tag{6'}$$

1. Ernst Potential

Next we rewrite the equations (5′) and (6′) in terms of matrix.

Let

$$G = \{g \in SL_3(\mathbf{C});\, g^*Jg = J\} \cong SU(1,2),$$

where $J = \begin{pmatrix} & & i \\ & 1 & \\ -i & & \end{pmatrix}$, and K its maximal compact subgroup, i.e.,

$$K = \{g \in G;\, g^*g = 1\}.$$

We define the Cartan involution Θ by $\Theta(g) = (g^*)^{-1}$ for $g \in G$.

Let $G = KAN$ be an Iwasawa decomposition with

$$A = \left\{ \begin{pmatrix} a & & \\ & 1 & \\ & & 1/a \end{pmatrix} ;\, a > 0 \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & & \\ v & 1 & \\ \psi + i|v|^2/2 & i\bar{v} & 1 \end{pmatrix} ;\, \psi \in \mathbf{R}, v \in \mathbf{C} \right\}.$$

Now we parametrize an element P in AN as follows [2]:

$$P = \begin{pmatrix} f^{1/2} & 0 & 0 \\ \sqrt{2}v & 1 & 0 \\ (\psi + i\,|v|^2)/f^{1/2} & \sqrt{2}i\bar{v}/f^{1/2} & 1/f^{1/2} \end{pmatrix}.$$

with f, v and ψ as above.

It is well known that (u, v) is a solution of (5′), (6′) if and only if P is a solution of the following equation:

$$d\,(\rho * dMM^{-1}) = 0 \qquad \text{with} \qquad M = \Theta(P)^{-1}P. \tag{7}$$

Let $\mathfrak{g}$ the Lie algebra of G, i.e.,

$$\mathfrak{g} = \{X \in sl_3(\mathbf{C});\, X^*J + JX = O\},$$

where J is as above. We denote by θ the involution of $\mathfrak{g}$ induced from the involution Θ of G.

Definition. *Let $\mathcal{A}$ and $\mathcal{I}$ be $\mathfrak{g}$-valued 1-forms defined by*

$$\mathcal{A} = \frac{1}{2}(dPP^{-1} + \theta(dPP^{-1})), \qquad \mathcal{I} = \frac{1}{2}(dPP^{-1} - \theta(dPP^{-1})).$$

We define a $\mathfrak{g}$-valued 1-form Ω with a spectral parameter to be

$$\Omega = \Omega(s) = \mathcal{A} + \frac{1 - 2sz - 2z\rho*}{\Lambda}\mathcal{I},$$

with $\Lambda = \{(1 - 2sz)^2 + 4s^2\rho^2\}^{1/2}$.

Note that $\Omega(0) = \mathcal{A} + \mathcal{I} = dPP^{-1}$.

PROPOSITION 2. *Ω satisfies the integrability condition, i.e.,*

$$d\Omega - \Omega \wedge \Omega = 0$$

if and only if P is a solution of (7).

For any solution P of the equation (7), by Proposition 2, there exists $\mathcal{P} = \mathcal{P}(s; z, \rho) \in SL(3, \mathbb{C}[[z, \rho, s]])$ which satisfies

$$d\mathcal{P} = \Omega\mathcal{P}, \qquad \mathcal{P}|_{s=0} = P$$

where $\mathbb{C}[[z, \rho, s]]$ is a ring of formal power series in z, ρ, s and $SL(3, \mathbb{C}[[z, \rho, s]])$ is a group consisting of all matrices of determinant 1 whose entries are the elements of $\mathbb{C}[[z, \rho, s]]$.

2. A Prescription for Constructing Solutions

Before giving a prescription for constructing solutions of the Ernst equations, we introduce a formal loop group and its subgroups, following [5].

Let $G^{(\infty)}$ be an infinite dimensional group

$$\{g(s) \in SL(3, \mathbb{C}[[s^{-1}]])\,; g(s)^* J g(s) = J\},$$

where $\mathbb{C}[[s^{-1}]]$ is a ring of formal power series in s^{-1} and $g(s)^* = {}^t\overline{g(\bar{s})}$.

Next we introduce a formal loop group $\mathcal{G}_R$. Let R be a ring of formal power series $\mathbb{C}[[z, \rho]]$ and I an ideal of R generated by ρ, i.e., $I = (\rho)$. We put

$$R_n = \begin{cases} I^n & \text{for} \quad n > 0 \\ R & \text{for} \quad n \le 0. \end{cases}$$

Then we define

$$\mathcal{G}_R = \{u = \sum_{n \in \mathbb{Z}} u_n t^n; u_n \in gl(3, R_n), u_0 \text{ is invertible}\},$$

and its subgroups

$$\mathcal{N}_R = \{u = \sum_{n \in \mathbb{Z}} u_n t^n \in \mathcal{G}_R\,; u_n = 0\,(n > 0), u_0 = 1\},$$

$$\mathcal{P}_R = \{u = \sum_{n \in \mathbb{Z}} u_n t^n \in \mathcal{G}_R\,; u_n = 0\,(n < 0)\}.$$

REMARK. *If we define*

$$\mathcal{G}_R^{(0)} = \{u = \sum_{n \in \mathbb{Z}} u_n t^n; u_n \in gl(3, R_{-n}), u_0 \text{ is invertible}\},$$

then $\mathcal{G}_R^{(0)}$ also forms a group. And for any $g(s) \in G^{(\infty)}$,

$$g((\frac{\rho}{t} + 2z - \rho t)^{-1}) \in \mathcal{G}_R \cap \mathcal{G}_R^{(0)}.$$

Our main theorem is:

THEOREM. *For any* $g(s) \in G^{(\infty)}$, *there exists uniquely an element* $k(t) \in \mathcal{G}_R$ *which satisfies the following conditions:*

(i) $\Theta(k(-\frac{1}{t})) = k(t), \det k(t) = 1$;

(ii) $k(t)g((\frac{\rho}{t} + 2z - \rho t)^{-1})^{-1}$ *is an element of* $\mathcal{P}_R$;

Putting $p(t) = k(t)g((\frac{\rho}{t} + 2z - \rho t)^{-1})^{-1} = \sum_{n\geq 0} p_n t^n$,

(iii) p_0 *is an element of* AN *and is a solution of the Ernst equation (7).*

For the proof we reduce the problem to Birkhoff decomposition (3.17) of formal loop groups established in [5]:

LEMMA. *Any element* u *of* $\mathcal{G}_R$ *can be uniquely decomposed as*

$$u = w^{-1}v, \quad w \in \mathcal{N}_R, \ v \in \mathcal{P}_R.$$

For the detail of the proof of the theorem, we refer to [3].

3. Examples of Solutions

In this section we shall see how the prescription given in the previous section works, giving some simple examples.

Note that $SL(2, \mathbb{R})$ can be embedded in G by the mapping

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & & b \\ & 1 & \\ c & & d \end{pmatrix}.$$

We use this embedding whenever we treat a field without electro-magnetic potentials.

Example 1 For $g(s) = \begin{pmatrix} 1 & 0 \\ -s^{-1} & 1 \end{pmatrix}$ with s^{-1} replaced by $s^{-1} = \frac{\rho}{t} + 2z - \rho t$, the element $k(t) \in \mathcal{G}_R$ in the theorem is determined in the following way: By the condition (i) of the theorem, $k(t)$ is written as

$$k(t) = \begin{pmatrix} a(-\frac{1}{t}) & b(t) \\ -b(-\frac{1}{t}) & a(t) \end{pmatrix},$$

so that

$$p(t) = \begin{pmatrix} a(-\frac{1}{t}) & b(t) \\ -b(-\frac{1}{t}) & a(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\rho}{t} + 2z - \rho t & 1 \end{pmatrix} \in \mathcal{P}_R. \tag{8}$$

Then the (1,2)-th entry of the right hand side of (8) can be expanded as

$$b(t) = b_1 t + b_2 t^2 + \cdots,$$

since p_0 is lower triangular.

In a similar way the (2.2)-th entry reads

$$a(t) = a_0 + a_1 t + a_2 t^2 + \cdots .$$

Since the (1,1)-th entry

$$\left(a_0 - \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots\right) + \left(b_1 t + b_2 t^2 + \cdots\right)\left(\frac{\rho}{t} + 2z - \rho t\right)$$

contains no negative-power-terms in t, it follows that $a(t) = a_0$.

By the same reason for the (2,1)-th entry, it follows that

$$b(t) = b_1 t, \quad \text{and} \quad b_1 + \rho a_0 = 0.$$

Since $\det k(t) = 1$, it follows that

$$a_0 = \frac{1}{\sqrt{1-\rho^2}}.$$

Therefore

$$p_0 = \frac{1}{\sqrt{1-\rho^2}} \begin{pmatrix} 1-\rho^2 & 0 \\ 2z & 1 \end{pmatrix},$$

and

$$M = \Theta(p_0^{-1}) p_0 = \frac{1}{1-\rho^2} \begin{pmatrix} (1-\rho^2)^2 + 4z^2 & 2z \\ 2z & 1 \end{pmatrix}.$$

This is the first example given in [4].

Next we give another example which has a non-trivial electro-magnetic potential.

Example 2 For $g(s) = \begin{pmatrix} 1 & & \\ cs^{-1} & 1 & \\ i|c|^2 s^{-2}/2 & i\bar{c}s^{-1} & 1 \end{pmatrix}^{-1}$ (where c is an arbitrary complex number), $k(t)$ is given by

$$k(t) = \begin{pmatrix} a & -\bar{c}\rho a t & -i|c|^2\rho^2 a t^2/2 \\ -2c\rho t^{-1}/(2-|c|^2\rho^2) & (2+|c|^2\rho^2)/(2-|c|^2\rho^2) & 2ic\rho t/(2-|c|^2\rho^2) \\ i|c|^2\rho^2 a t^{-2}/2 & -i\bar{c}\rho a t^{-1} & a \end{pmatrix},$$

and $M = \Theta(p_0^{-1}) p_0$ is given by

$$M = \begin{pmatrix} a^{-2} + 4|c|^2 z^2 + 4a^2|c|^4 z^4 & 2\bar{c}z + 4a^2\bar{c}|c|^2 z^3 & -2ia^2|c|^2 z^2 \\ 2cz + 4a^2 c|c|^2 z^3 & 1 + 4a^2|c|^2 z^2 & -2ia^2 cz \\ 2ia^2|c|^2 z^2 & 2ia^2\bar{c}z & a^2 \end{pmatrix}$$

where

$$a = \frac{2}{2-|c|^2\rho^2}.$$

References

[1]. P. Breitenlohner and D. Maison, *On the Geroch group*, Ann. Inst. Henri Poincaré **46** (1987), 215–246.

[2]. M.Gürses and B.C.Xanthopoulos, *Axially symmetric, static self-dual SU(3) gauge fields and stationary Einstein-Maxwell metrics*, Phys. Rev. D **26** (1982), 1912–1915.

[3]. T.Hashimoto and R.Sawae, *A linearization of* $S(U(1)\times U(2))\backslash SU(1,2)$ *σ-model*, to appear in Hiroshima. Math. J..

[4]. K. Nagatomo, *The Ernst equation as a motion on a universal Grassmann manifold*, Commun. Math. Phys. **122** (1989), 423–453.

[5]. K. Takasaki, *A new approach to the self-dual Yang-Mills equations II*, Saitama Math. J. **3** (1985), 11–40.